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Stress work effects in laminar flat-plate natural convection

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The effects of both viscous and pressure stress work are considered in laminar natural convection on flat surfaces. Whereas in previous investigations only viscous stress work effects were studied, it is established here that pressure work effects are generally rather more important both for gases and liquids. Variations in the properties of the fluid outside the natural convection boundary layer are shown to occur inevitably in this problem and are found to lead to effects comparable with those produced by pressure work. Property variations in the boundary layer itself are also found to be of importance. Such variations, both inside and outside the boundary layer, have been ignored in previous studies. Examples of flows are discussed in which the fluids are either perfect gases or fluids undergoing small changes in temperature and pressure. For both of these fluids, the effects of conditions of either constant temperature or constant entropy in the fluid outside the boundary layer are examined.

1. Introduction

A general statement of energy conservation for a viscous, heat conducting, compressible fluid can be deduced (Howarth 1953) by the balance of the rates of convection of specific internal and kinetic energies into an elementary volume of fluid compared with the rate of heat conduction into the volume and the rates of working of the body force, the pressure and the viscous stresses on the volume. Numerous alternative forms of the energy equation follow from this statement and are obtained by replacing internal energy by some other desired property of state. Furthermore, the kinetic energy term usually is eliminated by appeal to the momentum equation. However, the point to be noted is that in all such alternative forms there remain explicitly only certain parts of the terms representing the rates of working of the pressure and viscous stresses. In the following discussion, such remaining parts will be referred to, somewhat loosely, as the effects of pressure and viscous stress work.

In the discussion and analysis of natural convection flows, such pressure and viscous stress work effects are generally ignored. However, the influence and importance of viscous stress work effects in laminar flows have been examined by Gebhart (1962) and, more recently, by Gebhart & Mollendorf (1969). Both of these investigations considered specifically flows over semi-infinite flat surfaces set parallel to the direction of the gravity vector. Whereas Gebhart

(1962) considered flows generated by plate surface temperatures which vary as powers of x (the distance along the plate surface from the leading edge), Gebhart & Mollendorf (1969) considered flows generated by plate surface temperatures which vary exponentially in x. In both of these investigations it was established that the importance of viscous stress work is determined by a length scale equal to $c_p/g\beta$, where β is the volumetric coefficient of thermal expansion. Since this length scale is usually extremely large for most fluids, it was shown that this led to the conclusion that viscous stress work effects are very slight in most situations. For example, in the case of constant surface temperature, Gebhart (1962) showed that viscous stress work effects are governed simply by the ratio of x and the above large length scale. It is easy to show that this ratio is the Eckert number for this flow.

In both of the above investigations the term appearing explicitly in the energy equation used which stems from the rate of working of the pressure is ignored. The effect of the inclusion of this term was not considered by Gebhart (1962), whereas in Gebhart & Mollendorf (1969) (and in Gebhart 1969) the view was expressed that such pressure work effects can be ignored for liquids and, as a result, the subsequent analysis proceeded with such fluids in mind. Because of the lack of a thorough discussion of this view, it is felt that the role of pressure work in natural convection warrants further study and this is undertaken here. It is shown that pressure work effects are at least as important as (and in most cases rather more important than) those of viscous work for both liquids and gases.

In the studies of both Gebhart (1962) and Gebhart & Mollendorf (1969) no account was taken of variations of state properties in the fluid outside the moving boundary layer. This neglect is in keeping with their use of the Boussinesq form of the boundary-layer equations. However, one of the prime requirements for the correct use of the Boussinesq approximation is that the vertical extent of the fluid under consideration must be much less than any scale height characteristic of fluid-property variations (see, for example, Spiegel & Veronis 1960). This requirement cannot be met in the present context because the hydrostatic condition will be shown to introduce under all circumstances the above scale height $c_p/g\beta$ as a length scale characterizing at least some of the relevant fluid-property variations.

In order to obtain some comparison with the results of previous investigations, only flat-plate laminar natural convection is discussed here. For this configuration, the paper sets out to show that pressure and viscous work effects, together with the effects of property variations in the stationary fluid outside the boundary layer, are all governed by the same length scale $c_p/g\beta$. Thus, one may not include viscous stress work effects and ignore the other two effects, particularly since it will be shown that the latter are generally rather more important than the former. However, it should be remembered that, except possibly for a natural convection flow which occurs over an exceptionally large development length x, all three effects will usually be of little significance.

In view of the above points, it appears to be inevitable that we consider the fluid to be of variable properties and that the necessary equations of state and transport properties be examined with some care. In the present investigation the fluids studied are taken to be either a thermally and calorically perfect gas or a general fluid undergoing small changes in temperature and pressure. Two cases of fluid-property variations outside the boundary layer are discussed: those for which the temperature and, alternatively, the entropy are constants. Together with these property variations, two reasonably practical plate surface temperature conditions are discussed.

2. Boundary-layer equations and transformations

We consider a steady, two-dimensional, laminar fluid flow at high Reynolds number over a semi-infinite flat surface. Cartesian co-ordinates (x, y) are chosen to lie along and normal to the surface, respectively, and (g_x, g_y) are the components of the gravity vector in the (x, y) directions, respectively. We shall assume throughout that $|g_y/g_x|$ is at most of order unity so that the pressure gradient normal to the surface is zero according to first-order boundary-layer theory. Thus, the surface may be assumed to be tilted but, if so, not too far from the vertical, so that flows over surfaces which are almost horizontal (Stewartson 1958) are excluded from the discussion.

The boundary-layer continuity and momentum equations are, in the usual notation,

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0, \tag{1}$$

$$\rho\frac{Du}{Dt} = \rho g_x - \frac{dp}{dx}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right). \tag{2}$$

Conditions exterior to the boundary layer (denoted by suffix e), at which the exterior fluid is at rest, yield

$$\rho_e g_x - dp/dx = 0. \tag{3}$$

A useful and entirely general form of the energy equation (Howarth 1953, p. 56) is, in Cartesian tensor notation,

$$\left.\begin{aligned} \rho c_p\frac{DT}{Dt} &= \frac{\partial}{\partial x_i}\left(k\frac{\partial T}{\partial x_i}\right)+T\beta\frac{Dp}{Dt}+\Phi, \\ \text{where}\qquad \beta &= -\rho^{-1}(\partial\rho/\partial T)_p \end{aligned}\right\} \tag{4}$$

and Φ is known as the viscous dissipation function. Whilst Φ represents that part of the viscous work necessary for the correct balance of energy in this particular form of the energy equation, the term $T\beta Dp/Dt$ represents the corresponding part of the pressure work. It is the latter term which is ignored by Gebhart (1962, 1969) and Gebhart & Mollendorf (1969). In the steady boundary layer this energy equation becomes

$$\rho c_p\frac{DT}{Dt} = \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)+T\beta u\frac{dp}{dx}+\mu\left(\frac{\partial u}{\partial y}\right)^2. \tag{5}$$

We define a stream function ψ such that

$$\rho u = \partial\psi/\partial y, \quad -\rho v = \partial\psi/\partial x \tag{6}$$

and a non-dimensional temperature difference

$$\theta = (T - T_e(x))/\Delta T(x), \quad \Delta T(x) = T_w(x) - T_e(x). \tag{7}$$

We change the independent variables (x, y) to (ζ, η), where (Sparrow & Gregg 1958)

$$\zeta = x, \quad \eta = \left(\frac{\rho_r^2(-g_x(\Delta T)_r)\,\beta_r}{4\mu_r^2 x}\right)^{\frac{1}{4}} \int_0^y \frac{\rho}{\rho_r}\,dy_1, \tag{8}$$

the suffix r refers to any convenient reference condition and $(\Delta T)_r$ is any convenient reference temperature difference. Equation (3) is used to eliminate the pressure-gradient term in (2) and (5) and, on writing

$$\psi(x, y) = [4^3\rho_r^2\mu_r^2(-g_x(\Delta T)_r)\,\beta_r x^3]^{\frac{1}{4}} f(\zeta, \eta), \tag{9}$$

the momentum and energy equations (2) and (5) become

$$\left(\frac{\rho\mu}{\rho_r\mu_r} f''\right)' + 3ff'' - 2f'^2 + \frac{(\rho_e/\rho) - 1}{\beta_r(\Delta T)_r} = 4\zeta\left(f'\frac{\partial f'}{\partial\zeta} - \frac{\partial f}{\partial\zeta} f''\right), \tag{10a}$$

$$f(\zeta, 0) = f'(\zeta, 0) = f'(\zeta, \infty) = 0; \tag{10b}$$

$$\frac{c_{pr}}{c_p}\left(\frac{c_p}{c_{p_r}}\frac{\rho\mu}{\rho_r\mu_r}\frac{\theta'}{Pr}\right)' + 3f\theta' - \frac{4\zeta}{\Delta T} f'\left(\frac{dT_e}{d\zeta} + \frac{\theta d\Delta T}{d\zeta}\right) + \frac{4\zeta}{L}\frac{c_{pr}}{c_p}\left[\frac{\rho\mu}{\rho_r\mu_r}\frac{(\Delta T)_r}{\Delta T} f''^2 - \frac{f'}{\Delta T}\frac{\rho_e}{\rho}\frac{T\beta}{\beta_r}\right]$$
$$= 4\zeta\left(f'\frac{\partial\theta}{\partial\zeta} - \frac{\partial f}{\partial\zeta}\theta'\right), \tag{10c}$$

$$\theta(\zeta, 0) = 1, \quad \theta(\zeta, \infty) = 0. \tag{10d}$$

Primes denote differentiation with respect to η and the parameter L (which has the dimensions of length) is defined as

$$L = c_{pr}/(-g_x\beta_r). \tag{11}$$

In (10c), the effects of viscous work and pressure work are found to be proportional to the first and second terms in square brackets, respectively. Whilst the first term is of order f''^2, the second term is of order $f'T_r/(\Delta T)_r$, so that it would appear that for both liquids and gases the effect of pressure work is not necessarily small in comparison with the effect of viscous work. Furthermore, both stress work terms are seen to be multiplied by ζ/L. Thus we note that the importance of the two stress work terms (relative to those terms representing convection and diffusion of heat) is determined largely by the nature of the T_e and ΔT variations. These points will be examined in detail in the following sections.

On a pedantic note, η can be re-written from (8) as

$$\eta = (\tfrac{1}{4})^{\frac{1}{4}}\frac{Re^{\frac{1}{2}}}{x}\int_0^y \frac{\rho}{\rho_r}\,dy_1. \tag{12}$$

Here,

$$Re = \rho_r u_f x/\mu_r \tag{13}$$

is a Reynolds number based on the characteristic free convection velocity u_f (Ostrach 1964), where

$$u_f^2 = -g_x\beta_r(\Delta T)_r x. \tag{14}$$

Defined thus, η fits conveniently within the canon of compressible boundary-layer theory. Furthermore, from (11) and (14),

$$\zeta/L = u_f^2/c_{pr}(\Delta T)_r, \tag{15}$$

which is the Eckert number based on u_f (Ostrach 1964).

Since $c_{pr}/g\beta_r$ typically has values of order 10^4 m and 10^6 m for air and water, respectively, at atmospheric temperature and pressure conditions, it is to be appreciated that in most cases of natural convection on flat plates $\zeta/L \ll 1$. Therefore, in the following study, we shall obtain solutions to (10) to first order in ζ/L only.

3. Equations of state and transport properties

The two equations of state to be used are, first, that for a thermally perfect gas and, second, that for a general fluid undergoing small changes in temperature and pressure. In the latter case, owing to the order of accuracy required in the solution of (10), it is found to be necessary to consider this equation of state in rather more detail than is usually the case in natural convection studies.

For the thermally perfect gas,

$$p \propto \rho T, \quad \beta = 1/T. \tag{16}$$

Since p is constant across any cross-section of the boundary layer, we obtain without any additional approximation

$$\frac{(\rho_e/\rho)-1}{\beta_r(\Delta T)_r} = \frac{\Delta T}{(\Delta T)_r}\theta\frac{T_r}{T_e}, \tag{17}$$

$$\frac{T}{\Delta T}\frac{\rho_e}{\rho}\frac{\beta}{\beta_r} = \frac{T_r}{\Delta T}+\theta\frac{T_r}{T_e}. \tag{18}$$

For the general fluid, we assume $\rho = \rho(T, p)$ and expand about the reference condition r to second order in small quantities. Let

$$\kappa = \rho^{-1}(\partial\rho/\partial p)_T \tag{19}$$

and
$$T' = \frac{T_e}{T_r}-1, \quad p' = \frac{p}{p_r}-1, \quad \frac{T-T_r}{T_r} = \frac{\Delta T}{T_r}\theta + T', \tag{20}$$

where T', p' and $\Delta T/T_r$ are all small compared with unity, and θ is at most of order unity in the boundary layer but $\theta \equiv 0$ in the exterior fluid. We shall find in the following section that the leading terms in expressions for T' and p' are of order ζ/L. Consequently, in view of the remarks made at the end of the previous section, we see that we need to obtain the relation corresponding to (17) to first order in T' and p'. Thus, we obtain

$$\frac{(\rho_e/\rho)-1}{\beta_r(\Delta T)_r} \simeq \frac{\Delta T}{(\Delta T)_r}\theta\left[1+\frac{\Delta T\theta}{T_r}\frac{\beta_r T_r}{2}\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r + O\left(\frac{\Delta T}{T_r}\right)^2 \right.$$
$$\left. + T'\left\{\left(\frac{T}{\beta}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r + O\left(\frac{\Delta T}{T_r}\right)\right\} - p'\left\{\left(\frac{p}{\beta}\left(\frac{\partial\kappa}{\partial T}\right)_p\right)_r + O\left(\frac{\Delta T}{T_r}\right)\right\}\right]. \tag{21}$$

Because of the form of (10), the relation corresponding to (18) need be obtained to zeroth order in T' and p' only. Since $\beta = \beta(T, p)$, we expand the latter to first order in small quantities and obtain, to zeroth order in T' and p',

$$\frac{T}{\Delta T}\frac{\rho_e}{\rho}\frac{\beta}{\beta_r} \simeq \frac{T_r}{\Delta T} + \theta\left\{1+\beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r\right\} + O\left(\frac{\Delta T}{T_r}\right). \qquad (22)$$

Note that there are three variables, T', p' and $\Delta T/T_r$, which have been used to obtain (21). In the square-bracketed term of this result, it is seen that a term of first order in $\Delta T/T_r$ has been included in addition to the first-order terms in T' and p'. There are two reasons for this, the first being that in practical circumstances it is likely that $\Delta T/T_r$ is at least as large as or larger than T' and p'. The second reason is that, for one form of $\Delta T/T_r$ to be studied presently, we shall obtain from this particular term in $\Delta T/T_r$ a significant term of order T'. In the considerably simpler analysis leading to (22), terms of order $\Delta T/T_r$ may be safely ignored since the right-hand side of this relation is likely to be dominated by the first term, $T_r/\Delta T$.

Although derived with liquids in mind, (21) and (22) apply to any fluid for which any one property of state is a unique function of any other two state properties. Since for a thermally perfect gas

$$\left(\frac{\partial\beta}{\partial T}\right)_p = -\frac{1}{T^2}, \quad \left(\frac{\partial\kappa}{\partial T}\right)_p \equiv 0, \qquad (23)$$

we see that now (21) reduces to (17), at least to the order of the known terms in (21), viz. (21) becomes

$$\frac{(\rho_e/\rho)-1}{\beta_r(\Delta T)_r} \simeq \frac{\Delta T}{(\Delta T)_r}\theta[1-T'] \simeq \frac{\Delta T}{(\Delta T)_r}\theta[1+T']^{-1} = \frac{\Delta T}{(\Delta T)_r}\theta\frac{T_r}{T_e}. \qquad (24)$$

Similarly, (22) reduces to (18) since, to zeroth order in ζ/L, $T_r/T_e = 1$. However, for a liquid such useful simplifications are not always possible. For example, for water at atmospheric pressure and 293 °K (Reynolds 1971, p. 641)

$$\beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r \simeq \beta_r T_r\left(\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r \simeq 15{\cdot}2. \qquad (25)$$

Except for T_e and ΔT variations, which will be discussed in the following section, the remaining terms to be determined in (10) involve μ, c_p and k. In describing these properties for a gas, we shall assume for simplicity that

$$\mu \propto T^{\omega}, \quad c_p = \text{constant}, \quad Pr = \text{constant}. \qquad (26)$$

Thus, without additional approximation, we obtain

$$\frac{\rho\mu}{\rho_r\mu_r} = \left(1+\frac{\Delta T}{T_r}\frac{T_r}{T_e}\theta\right)^{\omega-1}\left(\frac{T_e}{T_r}\right)^{\omega-1}\frac{p}{p_r}. \qquad (27)$$

For the general fluid undergoing small changes in properties, we assume that $\mu = \mu(T)$, $c_p = c_p(T)$ and $k = k(T)$ and, in keeping with the approach resulting in (21) and (22), expand about the reference condition r to first order in small quantities. Let

$$a = \left(\frac{T}{\mu}\frac{d\mu}{dT}\right)_r, \quad b = \left(\frac{T}{c_p}\frac{dc_p}{dT}\right)_r, \quad c = \left(\frac{T}{k}\frac{dk}{dT}\right)_r. \qquad (28)$$

To first order in T' and p' we obtain

$$\left.\begin{aligned}
\frac{\rho\mu}{\rho_r\mu_r} &\simeq 1-\frac{\Delta T}{T_r}\theta(\beta_r T_r-a)+O\left(\frac{\Delta T}{T_r}\right)^2-T'\left\{\beta_r T_r-a+O\left(\frac{\Delta T}{T_r}\right)\right\}\\
&\qquad +p'\left\{\kappa_r p_r+O\left(\frac{\Delta T}{T_r}\right)\right\},\\
\frac{c_{pr}}{c_p} &\simeq 1-\frac{\Delta T}{T_r}b\theta+O\left(\frac{\Delta T}{T_r}\right)^2-T'\left\{b+O\left(\frac{\Delta T}{T_r}\right)\right\},\\
\frac{c_p}{c_{pr}}\frac{\rho\mu}{\rho_r\mu_r}\frac{1}{Pr} &\simeq \frac{1}{Pr_r}\left[1-\frac{\Delta T}{T_r}\theta(\beta_r T_r-c)+O\left(\frac{\Delta T}{T_r}\right)^2-T'\left\{\beta_r T_r-c+O\left(\frac{\Delta T}{T_r}\right)\right\}\right.\\
&\qquad \left.+p'\left\{\kappa_r p_r+O\left(\frac{\Delta T}{T_r}\right)\right\}\right].
\end{aligned}\right\} \quad (29)$$

4. Exterior conditions and surface temperature variations

As mentioned earlier, exterior property variations contribute terms in (10) which can be comparable with, or even greater than, the stress work terms in the energy equation (10*c*). Two simple cases are studied here of representative exterior property variations which contribute effects comparable with those of stress work. The cases are those for which, first, temperature and, second, entropy are constants. Thermally and calorically perfect gases and general fluids undergoing small changes in state are considered for each of these two cases.

For convenience in the following analysis, we write

$$X = \zeta/L. \quad (30)$$

The reference conditions r are taken to be those at the plate leading edge in the exterior fluid.

We consider first the case of the thermally and calorically perfect gas. Using the hydrostatic condition (3), together with the equation of state (16), we obtain for the two cases

$$\frac{T_e}{T_r}=1,\quad \frac{p}{p_r}=\frac{\rho_e}{\rho_r}=\exp-\frac{\gamma}{\gamma-1}X \quad \text{for} \quad T_e=\text{constant}, \quad (31)$$

$$\frac{T_e}{T_r}=1-X,\quad \frac{\rho_e}{\rho_r}=(1-X)^{1/(\gamma-1)},\quad \frac{p}{p_r}=\left(\frac{\rho_e}{\rho_r}\right)^{\gamma} \quad \text{for} \quad s_e=\text{constant}. \quad (32)$$

In order to obtain the results (32), use has been made of the thermodynamic relation

$$T\,ds = c_p\,dT-\beta T\,dp/\rho. \quad (33)$$

The results (31) and (32) for gases are valid for all values of X. Such accuracy may not be achieved for more general fluids. However, we recall that we require exterior property variations to first order in X only. Thus, on repeating the procedure used earlier to obtain (21), together with the use of (3) and (20), we obtain for a general fluid undergoing small changes in state

$$T'\equiv 0,\quad \kappa_r p_r p'\simeq \rho'\simeq -\alpha X+O(X^2) \quad \text{for} \quad T_e=\text{constant}, \quad (34)$$

$$\left.\begin{aligned}
&T'\simeq -X+O(X^2),\quad \rho'\simeq(\beta_r T_r-\alpha)X+O(X^2),\\
&\kappa_r p_r p'\simeq -\alpha X+O(X^2) \quad \text{for} \quad s_e=\text{constant}.
\end{aligned}\right\} \quad (35)$$

Here $$\rho' = \rho_e/\rho_r - 1, \quad \alpha = (\kappa p c_p/\beta)_r. \tag{36}$$

For the calorically and thermally perfect gas $\alpha = \gamma/(\gamma - 1)$, so that for this fluid, to order X, equations (31) and (32), and (34) and (35) are in agreement.

The two cases considered so far of constant exterior temperature and constant exterior entropy result in exterior property variations which are governed solely by the length scale L. This general characteristic will be obtained whenever one property of state is chosen to be constant, since it is then the hydrostatic condition (3) which is alone capable of introducing a length scale (in this case L) into the problem. The remaining relationship needed to specify the exterior property variations is provided by the purely thermodynamic relationship between two properties of state which results from the choice of a third state property being a constant. As an alternative to the above constant state property approach, the exterior property variations can be determined if one of them is specified. Thus, for example, a temperature stratification may be chosen as a function of ζ/l (where the length scale l need not be of the same order as L). For the more likely practical situation in which $l \ll L$, we see from (10) that temperature stratification effects will dominate completely the stress work effects. On the other hand, should the unlikely circumstances arise in which $l \gg L$, the hydrostatic condition (3) will still introduce L as a scale height for pressure (although p will depend on l as well now), thus affecting (10) to order ζ/L through equations such as (27) and (29). Thus, the important point emerges that, unless imposed temperature stratifications completely dominate stress work, there will in general be at least an exterior pressure variation which will produce effects comparable with those of stress work and which must be included in any considerations of stress work.

A point concerning the stability of the possible forms of exterior property variations should be mentioned. Stability criteria require (see, for example, Landau & Lifshitz 1959, p. 9) that the entropy increases with altitude. The above constant entropy fluids are therefore neutrally stable and it is easy to show that the calorically and thermally perfect gas at constant temperature is stable. No such general result can be demonstrated for the general fluid. There remains a number of other common functions of state which might be subjected to the constant state property approach. Pressure being constant must be excluded since the hydrostatic condition (3) ensures a pressure variation. Density might be held constant, but it is easy to show that for the calorically and thermally perfect gas this leads to instability. Consequently, conditions of constant temperature and constant entropy appear reasonably satisfactory from the stability viewpoint. Furthermore, since exterior property variations have been shown to occur inevitably within the context of this stress work investigation, these two cases have the advantage of not overly complicating the investigation by the introduction of a further length scale l.

In addition to the exterior property variations discussed above, two simple and representative cases of surface temperature variations are considered. These are, first, a surface temperature variation which results in ΔT being maintained constant and, second, a constant surface temperature. For these two cases

$$\Delta T/T_r = (\Delta T)_r/T_r \quad \text{for} \quad \Delta T = \text{constant} \tag{37}$$

and
$$\frac{\Delta T}{T_r} = \frac{(\Delta T)_r}{T_r} - \left(\frac{T_e}{T_r} - 1\right) \quad \text{for} \quad T_w = \text{constant}. \tag{38}$$

Thus for T_e constant ΔT is constant, whereas when s_e is constant, both of the above ΔT variations must be considered.

5. Expansions for $X \ll 1$

Following the discussions of the previous section, three cases of exterior properties and ΔT variations are to be considered for both perfect gases and general fluids. These are

$$\left.\begin{array}{ll} \text{(i)} & T_e = T_r = \text{constant}, \quad T_w = \text{constant, so that } \Delta T = (\Delta T)_r = \text{constant}, \\ \text{(ii)} & s_e = \text{constant}, \quad \Delta T = (\Delta T)_r = \text{constant}, \\ \text{(iii)} & s_e = \text{constant}, \quad T_w = \text{constant, so that } \Delta T = (\Delta T)_r - T_r([T_e/T_r] - 1). \end{array}\right\} \tag{39}$$

Since we have chosen for simplicity to consider exterior properties and ΔT variations which are governed by the single length scale L (which is also the length scale governing stress work) we see that (10) can be written entirely in terms of the independent variables (η, X), the latter variable being generally small compared with unity.

We shall consider the algebraically simpler case of gases first. We expand the dependent variables f and θ in the forms

$$\left.\begin{aligned} f(\eta, X) &= f_0(\eta) + X f_1(\eta) + O(X^2), \\ \theta(\eta, X) &= \theta_0(\eta) + X\theta_1(\eta) + O(X^2). \end{aligned}\right\} \tag{40}$$

Expressions for the additional dependent variables contained in (10) are found from (17), (18) and (27) to zeroth and first orders in X as required. We obtain

$$\frac{(\rho_e/\rho) - 1}{\beta_r(\Delta T)_r} = \theta_0 + X(\theta_1 + A(\eta)) + O(X^2), \tag{41a}$$

$$\rho\mu/\rho_r\mu_r = B(\eta)\,(1 - XC(\eta) + O(X^2)), \tag{41b}$$

$$\frac{1}{\Delta T}\left(\frac{\rho_e}{\rho}\frac{T\beta}{\beta_r} + \frac{dT_e}{dX} + \theta\frac{d\Delta T}{dX}\right) = \theta_0 + D(\eta) + O(X), \tag{41c}$$

$$(\Delta T)_r/\Delta T = 1 + O(X), \tag{41d}$$

where
$$B(\eta) = [1 + \{(\Delta T)_r/T_r\}\theta_0]^{\omega - 1}. \tag{41e}$$

The dependent variables A, C and D are given for the three cases in table 1 below.

On using the above expansions, (10) reduce to

$$\left.\begin{aligned} &(Bf_0'')' + 3f_0f_0'' - 2f_{00}'^{2} + \theta_0 = 0, \quad f_0(0) = f_0'(0) = f_0'(\infty) = 0; \\ &(B\theta_0')'/Pr + 3f_0\theta_0' = 0, \quad \theta_0(0) = 1, \quad \theta_0(\infty) = 0; \end{aligned}\right\} \tag{42}$$

$$\left.\begin{aligned} (Bf_1'' - BCf_0'')' + 3f_0f_1'' + 7f_1f_0'' - 8f_0'f_1' + \theta_1 + A = 0, \quad f_1(0) = f_1'(0) = f_1'(\infty) = 0; \\ Pr^{-1}(B\theta_1' - BC\theta_0')' + 3f_0\theta_1' + 7f_1\theta_0' - 4f_0'\theta_1 - [4f_0'(\theta_0 + D)] + 4Bf_0''^{2} = 0, \\ \theta_1(0) = \theta_1(\infty) = 0. \end{aligned}\right\} \tag{43}$$

	(i)	(ii)	(iii)
A	0	θ_0	$\theta_0(1+T_r/(\Delta T)_r)$
C	$\dfrac{\gamma}{\gamma-1}+(1-\omega)\dfrac{\dfrac{(\Delta T)_r}{T_r}\theta_1}{1+\dfrac{(\Delta T)_r}{T_r}\theta_0}$	$\dfrac{\gamma}{\gamma-1}+(1-\omega)\left[\dfrac{\dfrac{(\Delta T)_r}{T_r}(\theta_0+\theta_1)}{1+\dfrac{(\Delta T)_r}{T_r}\theta_0}-1\right]$	$\dfrac{\gamma}{\gamma-1}+(1-\omega)\left[\dfrac{\dfrac{(\Delta T)_r}{T_r}\left\{\theta_0\left(1+\dfrac{T_r}{(\Delta T)_r}\right)+\theta_1\right\}}{1+\dfrac{(\Delta T)_r}{T_r}\theta_0}-1\right]$
D	$T_r/(\Delta T)_r$	0	$(T_r/(\Delta T)_r)\,\theta_0$

TABLE 1

Equations (42) and (43) reduce† to those of Gebhart (1962) (his isothermal surface case) if the requirements $B = 1$ and $A = C = 0$ are met and if the term in square brackets in the equation (43) for θ_1 is ignored. With the approximation that $\omega = 1$ the requirement that $B = 1$ is met but, even with this rather restrictive approximation, the further requirements that $A = C = 0$ cannot be obtained (see table 1). It will be appreciated that the finite values of the variables A and C arise because of variations in the exterior and plate surface conditions and these effects are ignored by Gebhart (1962). According to the discussions of the previous section, we see that it is not possible to choose a combination of exterior conditions such that both A and C are zero simultaneously. The term in square brackets in equation (43) for θ_1 occurs because of the combination of pressure work effects (contributing an amount $4f_0'(\theta_0+T_r/(\Delta T)_r)$) and exterior and plate surface variations (contributing an amount $4f_0'(D-T_r/(\Delta T)_r)$). Together with the latter effect, pressure work effects are neglected also in Gebhart's analysis. It is seen from table 1 that, in most cases, the values of A, C and D (and the pressure work term) are not small compared with viscous work effects.

Inherent in both the analysis of Gebhart (1962) and that of Gebhart & Mollendorf (1969) is the assumption that $(\Delta T)_r/T_r \ll 1$, the Boussinesq form of the boundary-layer equations being used as a starting-point for their analyses. In order to obtain a further comparison with Gebhart's (1962) analysis, we shall also make this assumption in the subsequent discussion. For convenience, we write

$$\epsilon = (\Delta T)_r/T_r. \tag{44}$$

Clearly f_0, θ_0 and B in (42) can be expanded in integer powers of ϵ, e.g.

$$f_0 = f_{00}+\epsilon f_{01}+O(\epsilon^2), \quad \text{etc.}$$

However, we see from table 1 that the equations (43) for f_1 and θ_1 contain terms involving ϵ^{-1}, ϵ^0, ϵ^1 and higher positive powers of ϵ. The terms in ϵ^{-1} arise primarily because of pressure work contributions and also because of the chosen surface temperature variations. In order to accommodate such terms in a formal expansion procedure, it is necessary to re-phrase (10) and the expansions (40) and (41) in terms of independent variables η and X/ϵ. In view of the very

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† A slight difference between the equation (43) for θ_1 and Gebhart's corresponding equation is explained by the fact that Gebhart carries out his expansions in terms of $4X$.

	(i)	(ii)	(iii)
A_0	0	0	θ_{00}
A_1	0	θ_{00}	$\theta_{00}+\theta_{01}$
C_1	$\dfrac{\gamma}{\gamma-1}+(1-\omega)\theta_{10}$	$\dfrac{\gamma}{\gamma-1}+(1-\omega)(\theta_{10}-1)$	$\dfrac{\gamma}{\gamma-1}+(1-\omega)(\theta_{00}+\theta_{10}-1)$
D_0	1	0	θ_{00}
D_1	0	0	θ_{01}

TABLE 2

large values of L mentioned earlier, in most practical situations $X/\epsilon \leqslant 1$. Consequently, we write

$$\left.\begin{aligned} f(\eta, X/\epsilon) &= f_{00}(\eta)+\epsilon f_{01}(\eta)+O(\epsilon^2)+(X/\epsilon)\{f_{10}(\eta)+\epsilon f_{11}(\eta)+O(\epsilon^2)\}+O((X/\epsilon)^2), \\ \theta(\eta, X/\epsilon) &= \theta_{00}(\eta)+\epsilon\theta_{01}(\eta)+O(\epsilon^2)+(X/\epsilon)\{\theta_{10}(\eta)+\epsilon\theta_{11}(\eta)+O(\epsilon^2)\}+O((X/\epsilon)^2). \end{aligned}\right\} \tag{45}$$

Similarly, the expansions (41) are replaced by

$$\frac{(\rho_e/\rho)-1}{\beta_r(\Delta T)_r} = \theta_{00}+\epsilon\theta_{01}+O(\epsilon^2)+\frac{X}{\epsilon}\{\theta_{10}+A_0(\eta)+\epsilon(\theta_{11}+A_1(\eta))+O(\epsilon^2)\}+O((X/\epsilon)^2), \tag{46a}$$

$$\rho\mu/\rho_r\mu_r = B(\eta)\left[1-(X/\epsilon)\{C_0(\eta)+\epsilon C_1(\eta)+O(\epsilon^2)\}+O((X/\epsilon)^2)\right], \tag{46b}$$

$$\frac{1}{\Delta T}\left(\epsilon\frac{\rho_e}{\rho}\frac{T\beta}{\beta_r}+\frac{dT_e}{d(X/\epsilon)}+\theta\frac{d(\Delta T)}{d(X/\epsilon)}\right) = D_0(\eta)+\epsilon(\theta_{00}+D_1(\eta))+O(\epsilon^2)+O(X/\epsilon), \tag{46c}$$

$$(\Delta T)_r/\Delta T = 1+O(X/\epsilon) \tag{46d}$$

and

$$B(\eta) = 1-\epsilon B_1(\eta)+O(\epsilon^2), \tag{46e}$$

where

$$B_1(\eta) = (1-\omega)\theta_{00}. \tag{46f}$$

In all cases it is found that $C_0 = 0$. The remaining dependent variables are listed in table 2 for each of the three cases. On using the above expansions, (10) reduce to

$$f_{00}'''+3f_{00}f_{00}''-2f_{00}'^2+\theta_{00} = 0, \quad \theta_{00}''/Pr+3f_{00}\theta_{00}' = 0, \tag{47}$$

$$\left.\begin{aligned} f_{01}'''-(B_1f_{00}'')'+3(f_{00}f_{01}''+f_{01}f_{00}'')-4f_{00}'f_{01}'+\theta_{01} &= 0, \\ \theta_{01}''/Pr-(B_1\theta_{00}')'/Pr+3(f_{01}\theta_{00}'+f_{00}\theta_{01}') &= 0, \end{aligned}\right\} \tag{48}$$

$$\left.\begin{aligned} f_{10}'''+3f_{00}f_{10}''+7f_{00}''f_{10}-8f_{00}'f_{10}'+\theta_{10}+A_0 &= 0, \\ \theta_{10}''/Pr+3f_{00}\theta_{10}'+7f_{10}\theta_{00}'-4f_{00}'\theta_{10}-4f_{00}'D_0 &= 0, \end{aligned}\right\} \tag{49}$$

$$\left.\begin{aligned} & f_{11}'''-(B_1f_{10}''+C_1f_{00}'')'+3(f_{00}f_{11}''+f_{10}''f_{01})+7(f_{00}''f_{11}+f_{10}f_{01}'')-8(f_{00}'f_{11}'+f_{01}'f_{10}') \\ & \qquad +\theta_{11}+A_1 = 0, \\ & \theta_{11}''/Pr-(B_1\theta_{10}'+C_1\theta_{00}')'/Pr+3(f_{00}\theta_{11}'+f_{01}\theta_{10}')+7(f_{11}\theta_{00}'+f_{10}\theta_{01}') \\ & \qquad -4(f_{00}'\theta_{11}+f_{01}'\theta_{10})+4f_{00}''^2-4f_{00}'(\theta_{00}+D_1)-4f_{01}'D_0 = 0, \end{aligned}\right\} \tag{50}$$

with boundary conditions

$$\left.\begin{aligned} & f_{ij}(0) = f_{ij}'(0) = f_{ij}'(\infty) = 0 \quad \text{for} \quad i,j = 0,\ 1,\ 2,\ \ldots, \\ & \theta_{00}(0) = 1, \quad \theta_{00}(\infty) = 0, \quad \theta_{ij}(0) = \theta_{ij}(\infty) = 0 \quad \text{for all other}\ i,j. \end{aligned}\right\} \tag{51}$$

	(i)	(ii)	(iii)
A_0	0	0	θ_{00}
A_1	$\left(\left(\frac{\alpha}{\kappa\beta}\left(\frac{\partial\kappa}{\partial T}\right)_p\right)_r + \beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r\theta_{10}\right)\theta_{00}$	$\left(\left(\frac{\alpha}{\kappa\beta}\left(\frac{\partial\kappa}{\partial T}\right)_p\right)_r-\left(\frac{T}{\beta}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r + \beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r\theta_{10}\right)\theta_{00}$	$\theta_{01}+\theta_{00}\left[\left(\frac{\alpha}{\kappa\beta}\left(\frac{\partial\kappa}{\partial T}\right)_p\right)_r-\left(\frac{T}{\beta}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r + \beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r(\theta_{00}+\theta_{10})\right]$
C_1	$\alpha+(\beta_r T_r-a)\,\theta_{10}$	$\alpha+(\beta_r T_r-a)\,(\theta_{10}-1)$	$\alpha+(\beta_r T_r-a)\,(\theta_{10}+\theta_{00}-1)$
G_1	$b\theta_{10}$	$b(\theta_{10}-1)$	$b(\theta_{10}+\theta_{00}-1)$
$\bar{C}_1$	$\alpha+(\beta_r T_r-c)\,\theta_{10}$	$\alpha+(\beta_r T_r-c)\,(\theta_{10}-1)$	$\alpha+(\beta_r T_r-c)\,(\theta_{10}+\theta_{00}-1)$
D_0	1	0	θ_{00}
D_1	$\beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r\theta_{00}$	$\beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r\theta_{00}$	$\beta_r T_r\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r\theta_{00}+\theta_{01}$

TABLE 3

In view of the expansion procedures used to obtain approximate equations of state for general fluids, it is necessary to adopt expansions of the forms (45), since once more pressure work is found to contribute terms of order X/ϵ in (10). However, using (21), (22) and (29) we find that, for the general fluid, in place of the expansions (46),

$$\frac{(\rho_e/\rho)-1}{\beta_r(\Delta T)_r} = \theta_{00}+\epsilon(\theta_{01}+R_1(\eta)\,\theta_{00})+O(\epsilon^2)+\frac{X}{\epsilon}\{\theta_{10}+A_0(\eta)+\epsilon(\theta_{11}+A_1(\eta)) + O(\epsilon^2)\}+O((X/\epsilon)^2), \tag{52a}$$

$$\rho\mu/\rho_r\mu_r = 1-\epsilon B_1(\eta)+O(\epsilon^2)-(X/\epsilon)\{C_0(\eta)+\epsilon C_1(\eta)+O(\epsilon^2)\}+O((X/\epsilon)^2), \tag{52b}$$

$$c_{pr}/c_p = 1-\epsilon F_1(\eta)+O(\epsilon^2)-(X/\epsilon)\{G_0(\eta)+\epsilon G_1(\eta)+O(\epsilon^2)\}+O((X/\epsilon)^2), \tag{52c}$$

$$\frac{c_p}{c_{pr}}\frac{\rho\mu}{\rho_r\mu_r}\frac{1}{Pr} = \frac{1}{Pr_r}\left[1-\epsilon\bar{B}_1(\eta)+O(\epsilon^2)-\frac{X}{\epsilon}\{\bar{C}_0(\eta)+\epsilon\bar{C}_1(\eta)+O(\epsilon^2)\}+O((X/\epsilon)^2)\right], \tag{52d}$$

$$\frac{1}{\Delta T}\left(\epsilon\frac{\rho_e}{\rho}\frac{T\beta}{\beta_r}+\frac{dT_e}{d(X/\epsilon)}+\theta\frac{d(\Delta T)}{d(X/\epsilon)}\right) = D_0(\eta)+\epsilon(\theta_{00}+D_1(\eta))+O(\epsilon^2)+O(X/\epsilon), \tag{52e}$$

$$(\Delta T)_r/\Delta T = 1+O(X/\epsilon), \tag{52f}$$

where

$$\left.\begin{aligned} R_1(\eta) &= \frac{\beta_r T_r}{2}\left(1+\frac{1}{\beta^2}\left(\frac{\partial\beta}{\partial T}\right)_p\right)_r\theta_{00}, \quad B_1(\eta) = (\beta_r T_r-a)\,\theta_{00},\\ F_1(\eta) &= b\theta_{00}, \quad \bar{B}_1(\eta) = (\beta_r T_r-c)\,\theta_{00}.\end{aligned}\right\} \tag{53}$$

In all cases it is found that $C_0 = G_0 = \bar{C}_0 = 0$. The remaining dependent variables are listed in table 3 for each of the three cases. For the general fluid, the above expansions used with (10) yield (47) and (49) unchanged (except that Pr_r replaces Pr), whereas (48) and (50) are replaced by

$$\left.\begin{aligned} f_{01}'''-(B_1 f_{00}'')'+3(f_{00}f_{01}''+f_{01}f_{00}'')-4f_{00}'f_{01}'+\theta_{01}+R_1\theta_{00} &= 0,\\ \frac{\theta_{01}''}{Pr_r}-\frac{(\bar{B}_1\theta_{00}')'}{Pr_r}-\frac{F_1}{Pr_r}\theta_{00}''+3(f_{01}\theta_{00}'+f_{00}\theta_{01}') &= 0,\end{aligned}\right\} \tag{54}$$

$$\left.\begin{aligned}&f'''_{11}-(B_1f''_{10}+C_1f''_{00})'+3(f_{00}f''_{11}+f_{01}f''_{10})+7(f''_{00}f_{11}+f_{10}f''_{01})-8(f'_{00}f'_{11}+f'_{01}f'_{10})\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\theta_{11}+A_1=0,\\&\frac{\theta''_{11}}{Pr_r}-\frac{(\bar{B}_1\theta'_{10}+\bar{C}_1\theta'_{00})'}{Pr_r}-\frac{G_1}{Pr_r}\theta''_{00}-\frac{F_1}{Pr_r}\theta''_{10}+3(f_{00}\theta'_{11}+f_{01}\theta'_{10})\\&\qquad+7(f_{11}\theta'_{00}+f_{10}\theta'_{01})-4(f'_{00}\theta_{11}+f'_{01}\theta_{10})+4f''^2_{00}-4f'_{00}(\theta_{00}+D_1-F_1)\\&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad-4f'_{01}D_0=0.\end{aligned}\right\}\tag{55}$$

The boundary conditions are again given by (51).

Equations (54) and (55) reduce to (48) and (50) in the case of a thermally and calorically perfect gas with transport properties specified by (26). This may be established by the use of (16) and (23) and the following identities: $a = c = \omega$, $b = 0$, $\alpha = \gamma/(\gamma-1)$.

Equations (47) are the familiar equations of Pohlhausen (1921). In the succeeding equations, pressure work effects, together with exterior and surface property variations acting only through the equations of state (to produce A_0 and D_0), contribute the forcing effects for the terms of order X/ϵ (i.e. f_{10} and θ_{10}). The remaining information necessary for the determination of these terms is provided solely by Pohlhausen's equations, so that the forms of the variations in the transport properties of the fluid are immaterial to this order. Compared with these effects, we see from (50) and (55) that viscous work effects (which contribute the term $4f''^2_0$ to the equations for θ_{11}) play a secondary role, contributing to terms of order X, i.e. f_{11} and θ_{11}. To this latter order, however, we find in addition to viscous work effects a complicated combination of effects due to pressure work and exterior and surface property variations as well as those due to property variations inside the boundary layer. Consequently, the equations (50) and (55) for f_{11} and θ_{11}, although of an order directly comparable with those deduced by Gebhart (1962), are rather more complicated than his. From the above discussion and an examination of the data given in tables 2 and 3, it appears to be extremely doubtful whether circumstances could be found for any fluid such that (50) and (55) could be reduced to the corresponding equations of Gebhart (1962).

6. Results

Flows in air and water have been studied by numerical calculation. The following data were assumed to hold for air at atmospheric temperature and pressure conditions:

$$\gamma = \tfrac{7}{5},\quad \omega = 0{\cdot}76,\quad Pr = 0{\cdot}72. \tag{56}$$

For water at atmospheric pressure and 293 °K, transport properties and c_p data have been obtained from Kaye & Laby (1956, pp. 36, 53, 56) whilst the data for β and κ have been taken from Reynolds (1971, p. 641). Thus, the values used for water properties are

$$\left.\begin{aligned}&Pr = 7{\cdot}0,\quad a = -7{\cdot}2,\quad b = -0{\cdot}042,\quad c = 0{\cdot}616,\\&\beta T = 6{\cdot}1\times10^{-2},\quad \alpha = 9{\cdot}2,\quad \frac{T}{\beta}\left(\frac{\partial\beta}{\partial T}\right)_p = 15{\cdot}13,\quad \frac{1}{\kappa\beta}\left(\frac{\partial\kappa}{\partial T}\right)_p = 0{\cdot}795\times10^{-6}.\end{aligned}\right\}\tag{57}$$

$(\Delta T)_r/T_r$		$f_0''(0)$	$-\theta_0'(0)$	$f_1''(0)$	$-\theta_1'(0)$
0·1	(i)	0·68459	0·51266	1·24045	0·73996
	(ii)	0·68459	0·51266	1·63063	0·77931
	(iii)	0·68459	0·51266	1·75519	0·88671
0·2	(i)	0·69245	0·52001	1·23923	0·80427
	(ii)	0·69245	0·52001	1·65623	0·79610
	(iii)	0·69245	0·52001	1·81446	0·94171

TABLE 4. $Pr = 0{\cdot}72$ (for other data see (56))

The Runge–Kutta–Merson method was used throughout the numerical integration procedures. Results of the solutions to (42) and (43) are given in table 4 for the three cases studied, property data being given by (56) and values of $(\Delta T)_r/T_r$ of 0·1 and 0·2 being selected. The property data of (56) were used again in the solutions of (47)–(50). Results of the latter solutions are given in table 5 and graphs of the perturbation functions of f' and θ are presented in figures 1 (*a*) and (*c*). The results of the solutions of (47), (49), (54) and (55), using the property data for water (equations (57)), are included also in table 5 whilst the corresponding graphs of the perturbation functions of f' and θ are given in figures 1 (*b*) and (*d*).

On comparing figures 1 (*a*) and (*c*) with figures 1 (*b*) and (*d*) one sees that the perturbation functions up to and including those of order X/ϵ (carrying subscripts 00, 01 and 10) show similar trends, the differences in relative magnitudes between the graphs for air and water being reflected in the differences in the relative magnitudes of the corresponding data of table 5. However, the graph of θ_{10}, case (i), for water (figure 1 *d*) shows a rather curious kink in it which is not present in the corresponding graph for air. In contrast, the data of order X (subscript 11) show considerable differences between the cases for the two fluids. Whilst the graphs of f'_{11} for air (figure 1 *a*) are, on the whole, the mirror image of those for water (figure 1 *b*), f'_{11}, case (iii), for water (figure 1 *b*) shows an unusual, rapid but small oscillation close to $\eta = 0$. The graphs of θ_{11} for both air (figure 1 *c*) and water (figure 1 *d*) have similar trends near $\eta = 0$ but for water θ_{11} changes sign at larger values of η, this behaviour not occurring in the cases for air. Without a considerable amount of numerical analysis it is not possible to determine the causes of these differences, although it seems likely that they lie in the differences in property variations between the two fluids.

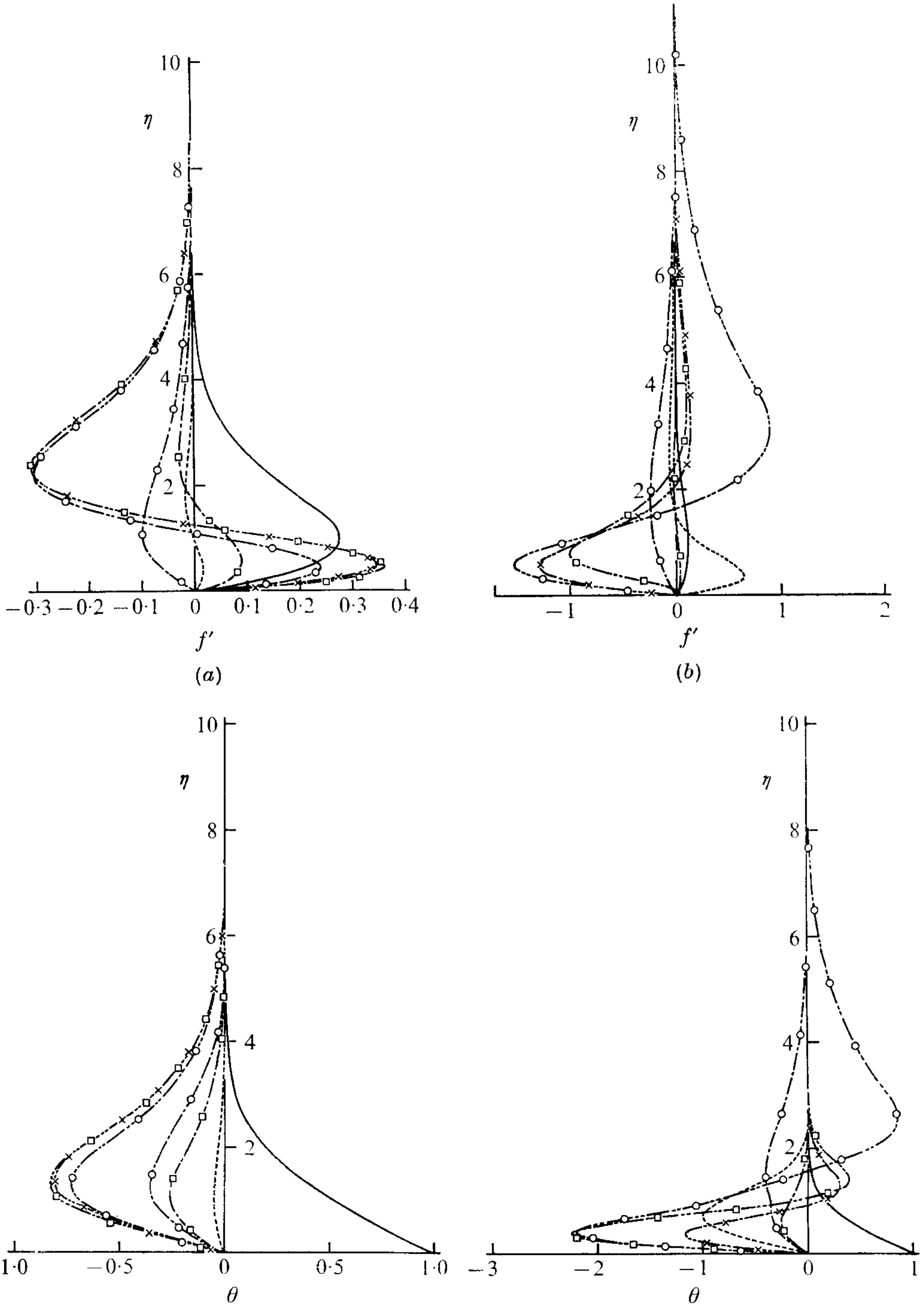
It will be seen in table 5 that, in both of the examples of case (ii) presented, both $f''_{10}(0)$ and $\theta'_{10}(0)$ are zero. In fact, the solutions of (49) for case (ii) conditions (in which the forcing terms A_0 and D_0 are both zero) are

$$f_{10} = \theta_{10} = 0. \tag{58}$$

This result holds true for all values of the Prandtl number, that is to say, no matter what the fluid. The term A_0 is zero whenever the temperature difference ΔT is constant (as it is in all cases (i) and (ii)) and D_0 is zero because the effects of pressure work and exterior temperature variation cancel exactly. The pressure work effect to this order depends only on the Prandtl number, being determined

Pr		$f''_{00}(0)$	$f''_{01}(0)$	$f''_{10}(0)$	$f''_{11}(0)$	$-\theta'_{00}(0)$	$-\theta'_{01}(0)$	$-\theta'_{10}(0)$	$-\theta'_{11}(0)$
0·72	(i)	0·67602	0·08973	−0·14929	1·22048	0·50463	0·08402	0·51440	0·77503
	(ii)	0·67602	0·08973	0	1·60204	0·50463	0·08402	0	0·76038
	(iii)	0·67602	0·08973	0·38839	1·74914	0·50463	0·08402	0·41847	0·90032
7	(i)	0·45078	4·26780	−0·28066	−5·27616	1·05431	2·00123	0·88207	9·16741
	(ii)	0·45078	4·26780	0	−5·42370	1·05431	2·00123	0	5·20135
	(iii)	0·45078	4·26780	0·25615	1·01587	1·05431	2·00123	0·80192	9·36746

TABLE 5. See (56) and (57) for further property data used for Pr equal to 0·72 and 7, respectively



(*a*) (*b*)

(*c*) (*d*)

FIGURE 1. (*a*) f' for $Pr = 0{\cdot}72$. (*b*) f' for $Pr = 7{\cdot}0$. (*c*) θ for $Pr = 0{\cdot}72$. (*d*) θ for $Pr = 7{\cdot}0$. ——, 00 perturbation; – – – –, 01 perturbation; — · — · —, 10 perturbation; — · · — · · —, 11 perturbation; ○, case (i); ×, case (ii); □, case (iii).

$Pr = 0{\cdot}72$ $\qquad$ $\theta'_{00}(0) = -0{\cdot}50463$ $\qquad$ $\theta'_{11}(0) = 0{\cdot}30024$

TABLE 6. Gebhart's (1962) results. Because Gebhart (1962) carried out his expansions corresponding to (45) in powers of $4X$, $\theta'_{11}(0)$ is obtained by multiplying Gebhart's function $\phi'_1(0)$ by 4.

by the solution of (47) alone. Thus it would appear that the condition of constant entropy is alone capable of creating an exterior temperature distribution such as to cause this cancellation. For example, it is easy to show that the only exterior temperature variation which will result in cancellation is, to order X, $dT'/dX = -1$. Now, this requirement, together with the use of the hydrostatic condition (3) and the definition of an unknown function of state $z = z(T, p)$, say, such that z be constant, results in the requirement that z must satisfy

$$\left(\frac{\partial z}{\partial p}\right)_T \left(\frac{\partial T}{\partial z}\right)_p = -\left(\frac{\partial T}{\partial p}\right)_z = -\frac{T\beta}{\rho c_p}. \tag{59}$$

Comparison of this result with (33) suggests that one may, in all circumstances, identify z with s.

Various simplifications to the description of property data for water at the above conditions (equations (57)) have been investigated. For example, if the terms in $(\alpha/\kappa\beta)(\partial\kappa/\partial T)_p$ are ignored (see table 3), the results for $f''_{11}(0)$ and $\theta'_{11}(0)$ are found to differ from those given in table 5 only in the fifth decimal place. Thus, whilst it appears to be permissible to ignore the effects of κ variations on terms in p' in (10), the same probably cannot be said for the terms in p' in (29) since the latter involve κ through α and here α is not small compared with unity. Another obvious possible simplification is to ignore all terms in $\beta T - c$ (see equation (53) for $\bar{B}_1$ and $\bar{C}_1$ in table 3). Compared with the results of table 5, this is found to result in changes of no more than 2 % in $f''_{01}(0)$ and $f''_{11}(0)$, the corresponding changes in $\theta'_{01}(0)$ and $\theta'_{11}(0)$ being no more than 17 and 4 %, respectively. Finally, acting on a suggestion of Maslen (1958), the effects of ignoring c_p and k variations with temperature have been examined. Setting $b = c = 0$ in (53) and table 3 is found to produce variations in $f''_{01}(0)$, $f''_{11}(0)$, $\theta'_{01}(0)$ and $\theta'_{11}(0)$ which are almost the same as those produced by setting $\beta T - c = 0$. Consequently, it would appear that such simplifications to βT, b and c are not realistic, at least as far as $\theta'_{01}(0)$ is concerned.

Gebhart's (1962) results for $Pr = 0{\cdot}72$ are quoted in table 6, where they have been recast in terms of the present variables. Whereas the result for $\theta'_{00}(0)$ is in agreement with the corresponding result of table 5, the value of $\theta'_{11}(0)$ inferred from Gebhart's calculations bears no comparison with the present results. Apart from it being difficult to say which of the three cases in table 5 one should compare with Gebhart's result, the latter has the wrong sign. No comparison can be made with Gebhart's (1962) graphical data since the latter are presented for a value of Pr of 100 only.

7. Conclusions

Stress work effects have been considered in detail in steady laminar natural convection flows on semi-infinite flat plates. In agreement with the results of Gebhart (1962), it is found that the significance of viscous work effects is determined by an Eckert number equal to $xg\beta/c_p$. However, contrary to the view of Gebhart (1962, 1969), it is found that the effects of pressure work are always rather more important than those of viscous work for both liquids and gases, since the significance of pressure work effects is found to depend on the ratio of the above Eckert number and $(\Delta T)_r/T_r$, the latter temperature ratio being less than unity. Thus, pressure work depends on the parameter $xg\beta T/(c_p \Delta T)$, so that the importance of pressure work relative to viscous work is increased when ΔT is small.

Exterior property variations in the stationary fluid surrounding the natural convection flow are found to be of importance in this problem and must be accounted for correctly. In the simplest possible situations with these property variations, in which a given function of state is assumed to be constant, it is found that the hydrostatic condition introduces exterior fluid-property variation effects which are directly comparable with those of pressure work.

Further conclusions to be drawn from the present investigation are that variations in μ, c_p and k must be included correctly and that the buoyancy term in the natural convection momentum equation must be determined to a higher order than is usually the case (cf. equations (10)). Since the conventional Boussinesq form of the boundary-layer equations (as used by Gebhart 1962; Gebhart & Mollendorf 1969) is incapable of including either exterior property variations or transport property variations, the use of this form of the equations must be avoided within the context of the present problem.

In the present investigation, one would have wished for more detailed methods of description of state and transport property variations than those used in (21) and (29). Of particular note here are the methods employed by Poots & Raggett (1967) and Poots & Miles (1967) in their studies of variable property effects in forced convection and film condensation, respectively. However, the use here of their high degree polynomial approximations to state and transport property data would have tended to obscure the importance of the parameter $\Delta T/T$, particularly in its role (in conjunction with the Eckert number) of describing pressure work.

Since the Eckert number $xg\beta/c_p$ is likely to be very small compared with unity in most practical circumstances, it follows that stress work effects will be of little significance. Usually, turbulent flow will have occurred well before the Eckert number has increased (with x) sufficiently for stress work to be of any importance. Furthermore, higher order boundary-layer effects (Yang & Jerger 1964), providing additional corrections to f and θ of order $Re^{-\frac{1}{2}}$ (Re defined in (13)), will usually dominate stress work effects. However, as Gebhart (1962) has remarked, for perfect gases ($\beta = 1/T$) at low temperatures, the Eckert number (and, therefore, the significance of stress work) is increased. Another situation in which stress work might be of more importance could occur near the critical point since both β and c_p are singular there. This aspect is undergoing further investigation.

So as to provide some comparison with Gebhart's (1962) results, the effects of stress work and the associated exterior fluid properties and transport property variations on the plate shear stress ($\propto f''(0)$) and heat transfer rate ($\propto \theta'(0)$) have been estimated for air and water at room temperature and pressure conditions.

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